Carleton UNIVERSITY
Canada's Capital University

Mathematics Tutorial Series

Differential Calculus #10

The Tangent and Secant functions + Quotient Rule

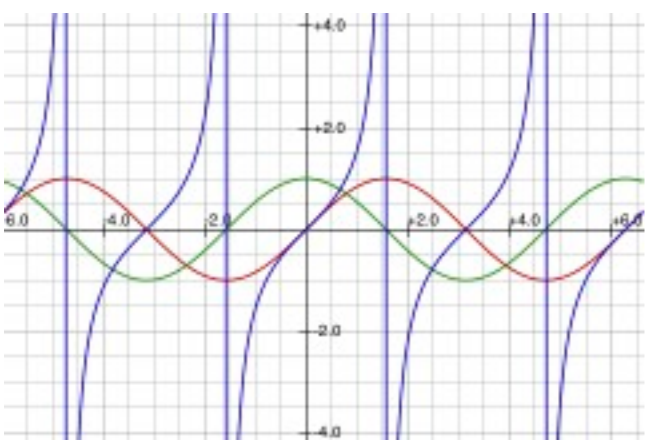
The tangent function is formally defined by

$$\tan x = \frac{\sin x}{\cos x}$$

Note: the word "tangent" is recycled here. We now have tangent lines and also tangent functions.

So $\tan x$ has value 0 whenever $\sin x$ does and has a vertical asymptote whenever $\cos x = 0$.

Here
$y = \tan x$ is **blue**,
$y = \sin x$ is **red** and
$y = \cos x$ is **green**.
The blue vertical lines indicate vertical asymptotes of the tangent function.



The function $\tan x$ is a periodic function that takes on every real value. This means that for every y you can find a value of x such that $\tan x = y$.

BUT ALSO we can always take $-\frac{\pi}{2} < x < \frac{\pi}{2}$

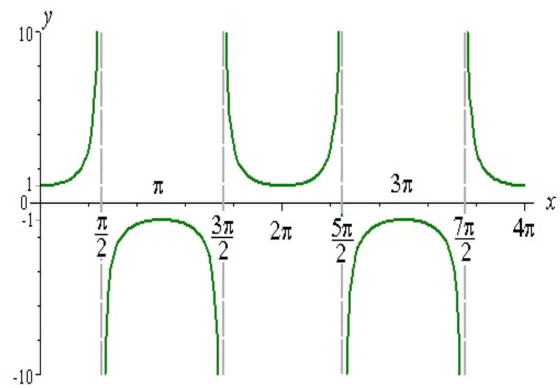
Hence $\tan x$ can be used in a model where we want to have a bounded set (domain) mapped to the whole real axis.

Definition: The **secant of x** is defined by

$$\sec x = \frac{1}{\cos x}$$

Since $cos\ x$ is always between -1 and $+\ 1$, the value of $sec\ x$ is always $\geq\ +1$ or $\leq\ -1$.

The graph is as follows:



Used mainly as a technical tool to express derivatives and integrals simply.

Quotient Rule for Differentiation

Suppose we want to find the derivative of a quotient

$$p(x) = \frac{f(x)}{g(x)}.$$

We can write this as:

$$P(x) = f(x) \cdot g(x)^{-1}$$

This is a product so we reach for the Product Rule.
Also the derivative of $g(x)^{-1}$ uses the Chain Rule.

$$(g(x)^{-1})' = -1 \cdot g(x)^{-2} \cdot g'(x)$$

[This is like the derivative of u^{-1} where $u\ =\ g(x)$]

All together

$$p' = f'g^{-1} + f \cdot (-1)g^{-2}g'$$

$$= \frac{f'g}{g^2} - \frac{fg'}{g^2}$$

$$= \frac{f'g - fg'}{g^2}$$

The Quotient Rule

$$\left(\frac{\boldsymbol{f}}{\boldsymbol{g}}\right)' = \frac{\boldsymbol{f'g - fg'}}{\boldsymbol{g^2}}$$

Derivative of *tan x*

The main example of the quotient rule is:

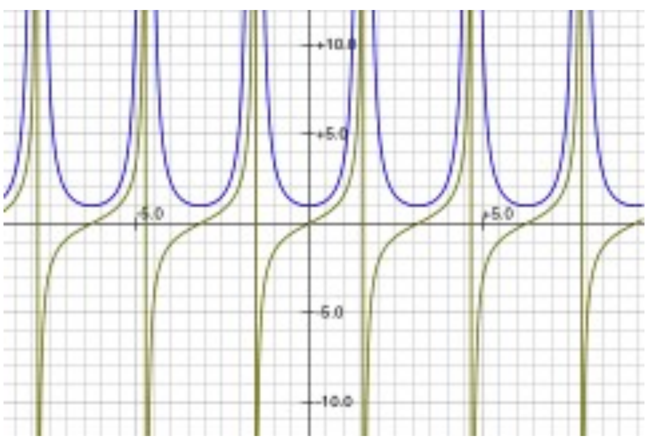
$$\tan x \ = \frac{\sin x}{\cos x}$$

$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)'$$

$$= \frac{\cos x \ \cdot \ \cos x - \sin x \ \cdot (-\sin \ x)}{\cos^2 x}$$

$$= \frac{cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

Hence $(\tan x)' = \sec^2 x$



Here $\tan x$ is green and $\sec^2 x$ is blue.

Derivative of $\boldsymbol{sec\ x}$

By definition $\sec x = (\cos x)^{-1}$.

Then, by the Quotient Rule, the derivative is:

$$(\sec x)' = (-1) \cdot (\cos x)^{-2} \cdot (\cos x)'$$

$$= (-1) \cdot (\cos x)^{-2} \cdot (-\sin x)$$

$$= \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x}$$

$$= \sec x \ \cdot \tan x$$

Hence $(\boldsymbol{sec\ x})' = \boldsymbol{sec\ x\ tan\ x}$

Summary

	Domain	Range
$Sin\ x$	x can be any real number	$-1 \leq y \leq +1$
$Cos\ x$	x can be any real number	$-1 \leq y \leq +1$
$Tan\ x$	x can be any real number <u>except</u> values making $cos\ x = 0$ So… Any x except odd multiples of $\frac{\pi}{2}$	y can be any real number

Derivatives of Trig Functions: If we measure x in radians then:

$$(\sin x)' = \cos x$$

$$(\cos x)' = -\sin x$$

$$(\tan x)' = sec^2 x$$

$$(\sec x)' = \sec x \tan x$$

The Quotient Rule

$$\left(\frac{\boldsymbol{f}}{\boldsymbol{g}}\right)' = \frac{\boldsymbol{f}'\boldsymbol{g} - \boldsymbol{f}\boldsymbol{g}'}{\boldsymbol{g}^2}$$